Some remarks on the interpretation of degree of nonextensivity

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Abstract

Recently we have demostrated that the nonextensitivity parameter q occuring in some applications of Tsallis statistics (known also as index of the corresponding Lévy distribution) is, in $q>1$ case, given entirely by the fluctuations of the parameters of the usual exponential distribution. We show here that this interpretation is valid also for the $q<1$ case. The parameter q is therefore a measure of fluctuations of the parameters of the usual exponential distribution.

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Recently we have demostrated that the nonextensitivity parameter q occuring in some applications of Tsallis statistics [1] (known also as index of the corresponding Lévy distribution) is, in $q > 1$ case, given entirely by the fluctuations of the parameters of the usual exponential distribution [2]. It means that:

- when in some exponential formula describing distribution of a quantity ε of physical interest:

$$L_{q=1}(\varepsilon) \,=\, C_{q=1}\, \exp\left[-\frac{\varepsilon}{\chi}\right], \tag{1}$$

  one allows the parameter χ to fluctuate around some mean value χ_0, and

- if these fluctuations are described by simple Gamma distribution of the form

$$f(\frac{1}{\chi}) \,=\, \frac{1}{\Gamma(\alpha)}\,\mu\,\left(\frac{\mu}{\chi}\right)^{\alpha-1}\,\exp\left(-\,\frac{\mu}{\chi}\right) \tag{2}$$

  depending on two parameters

$$\alpha \,=\, \frac{1}{q-1} \qquad \text{and} \qquad \mu = \alpha\chi_0, \tag{3}$$

- then, as result, one gets the following power-like distribution for the quantity ε of interest:

$$L_q(\varepsilon) \,=\, C_q\,\left[1\,-\,(1\,-\,q)\,\frac{\varepsilon}{\chi_0}\right]^{\frac{1}{1-q}}, \tag{4}$$

  known also as Lévy distribution with index q, where

$$q \,=\, 1\,+\,\frac{\left\langle\left(\frac{1}{\chi}\right)^2\right\rangle\,-\,\left\langle\frac{1}{\chi}\right\rangle^2}{\left\langle\frac{1}{\chi}\right\rangle^2}, \tag{5}$$

  i.e., where it is entirely given by the relative variance of the parameter $1/\chi$ of the initial distribution (1) ($<\,...\,>$ denotes the corresponding averages with respect to distribution $f(\chi)$).

The proof presented in [2] was limited to the $q > 1$ case and the physical discussion provided there was also concentrated on such situation. Because q can be also interpreted as the so called nonextensivity parameter occuring in some applications of Tsallis statistic [1], it would be interesting to check if such interpretation can be extended to the $q < 1$ case as well. We shall demonstrate below that this is indeed the case [3].

The essential difference between these two cases is, for the purpose of present discussion, that whereas for $q > 1$ probability distribution $L_q(\varepsilon)$ is well defined for the whole range of variable ε, $\varepsilon \in (0, \infty)$, for $q < 1$ it is defined only for $\varepsilon \in [0, \chi_0/(1-q)]$. As it was done in [2] we shall deduce the form of function $f(1/\chi)$, describing fluctuations in χ, which would lead from the exponential distribution $L_{q=1}$ to the power-like Lévy distribution $L_{q<1}$

$$L_{q<1}(\varepsilon;\chi_0) \,=\, C_q \left[1 \,-\, \frac{\varepsilon}{\alpha'\chi_0}\right]^{\alpha'} \,=\, C_q \int_0^{\infty} \exp\left(-\frac{\varepsilon}{\chi}\right) f\left(\frac{1}{\chi}\right) d\left(\frac{1}{\chi}\right) \tag{6}$$

(for simplicity we denote $\alpha' = \frac{1}{1-q}$). From the representation of the Euler gamma function we have

$$\left[1 \,-\, \frac{\varepsilon}{\alpha'\chi_0}\right]^{\alpha'} \,=\, \left(\frac{\alpha'\chi_0}{\alpha'\chi_0 - \varepsilon}\right)^{-\alpha'} \,=\, \frac{1}{\Gamma(\alpha')} \int_0^{\infty} d\eta\, \eta^{\alpha'-1} \exp\left[-\eta\, \left(1 \,+\, \frac{\varepsilon}{\alpha'\chi_0 - \varepsilon}\right)\right]. \tag{7}$$

Changing now variables under the integral in such a way that $\chi \,=\, \frac{\alpha'\chi_0-\varepsilon}{\eta}$ one immediately obtains Eq. (6) with $f(1/\chi)$ given by the following gamma distribution

$$f\left(\frac{1}{\chi}\right) \,=\, \frac{1}{\Gamma(\alpha')}\, (\alpha'\chi_0 - \varepsilon) \left(\frac{\alpha'\chi_0 - \varepsilon}{\chi}\right)^{\alpha'-1} \exp\left(-\frac{\alpha'\chi_0 - \varepsilon}{\chi}\right) \tag{8}$$

with parameters α' and $\mu(\varepsilon) = \alpha'\chi_0 - \varepsilon$. This time, contrary to the $q > 1$ case of [2], fluctuations depend on the value of the variable in question, i.e., the mean value and variance are both ε-dependent:

$$\left\langle\frac{1}{\chi}\right\rangle \,=\, \frac{1}{\chi_0 - \frac{\varepsilon}{\alpha'}} \qquad \text{and} \qquad \left\langle\left(\frac{1}{\chi}\right)^2\right\rangle - \left\langle\frac{1}{\chi}\right\rangle^2 \,=\, \frac{1}{\alpha'} \cdot \frac{1}{\left(\chi_0 - \frac{\varepsilon}{\alpha'}\right)^2}. \tag{9}$$

However, the relative variance

$$\omega \,=\, \frac{\left\langle\left(\frac{1}{\chi}\right)^2\right\rangle - \left\langle\frac{1}{\chi}\right\rangle^2}{\left\langle\frac{1}{\chi}\right\rangle^2} \,=\, \frac{1}{\alpha'} \,=\, 1 \,-\, q \tag{10}$$

remains ε-independent (exactly like in the case of $q > 1$) and depends only on parameter q. It means therefore that the parameter q in Lévy distribution $L_q(\varepsilon)$ describes the relative variance of fluctuations of parameter χ in $L_{q=1}(\varepsilon)$ for all values of q (both for $q > 1$, where $\omega = q - 1$, cf. [2] and for $q < 1$ as given above, where $\omega = 1 - q$).

In [2] we have proposed a general explanation of the meaning of function $f(\chi)$ describing fluctuations of some variable χ. The question one is interested in is

why, and under what circumstances, it is the gamma distribution that describes fluctuations. To this end we have started with general Langevin type equation [4] for the variable χ

$$\frac{d\chi}{dt} + \left[\frac{1}{\tau} + \xi(t)\right]\chi = \phi = \mathrm{const} > 0 \tag{11}$$

(with damping constant τ and source term ϕ). For stochastic processes defined by the white gaussian noise form of $\xi(t)$ (cf. [2] for details) it can be shown that distribution function for the variable χ satisfies the Fokker-Planck equation ($K_{1,2}$ are the corresponding intensity coefficients, cf. [2])

$$\frac{df(\chi)}{dt} = -\frac{\partial}{\partial\chi}K_1\, f(\chi) + \frac{1}{2}\frac{\partial^2}{\partial\chi^2}K_2\, f(\chi), \tag{12}$$

i.e., it is indeed given by the Gamma distribution in variable $1/\chi$ of the form (2) with $\mu = \alpha\chi_0$. Notice that it differs from Eq. (8) only in the form of parameter μ, which in (8) depends also on the physical quantity of interest ε.

As an illustration of the genesis of Eq. (11) we have discussed in [2] the case of fluctuations of temperature (i.e., the situation where $\chi = T$) [5]. Suppose that we have a thermodynamic system, in a small (mentally separated) part of which the temperature fluctuates around some mean value T_0 (which can be also understood as an equilibrium temperature) with $\Delta T \sim T$. The unevitable exchange of heat between this selected region and the rest of the system is described by Eq. (11) in which

$$\phi = \phi_{q<1} = \frac{1}{\tau}\left(T_0 - \frac{\varepsilon}{\alpha'}\right) \qquad \text{whereas} \qquad \phi = \phi_{q>1} = \frac{T_0}{\tau}. \tag{13}$$

It means that the corresponding process of heat conductivity is, for $q < 1$ case, described by the following equation (here $T' = T_0 - \tau\xi(t)T$)

$$\frac{\partial T}{\partial t} - \frac{1}{\tau}(T' - T) + \frac{\varepsilon}{\tau\alpha'} = 0, \tag{14}$$

which differs from the corresponding equation for $q > 1$ case only by the last term describing the presence the internal heat source. It has a sense of dissipative transfer of energy from the region where (due to fluctuation) we have higher T. It could be any kind of convection type flow of energy, for example it could be connected with emission of particles from that region. The heat release given by $\varepsilon/(\tau\alpha')$ depends on ε (but it is only a part of ε, which is released). In the case of such energy release (connected with emission of particles) there is additionale cooling of the whole system. If this process is sufficiently fast, it could happen that there is no way to reach a stationary distribution of temperature (because the transfer of the heat from

the outside can be not sufficient for development of the state of equilibrium). On the other hand (albeit this is not our case here) for the reverse process we could face the "heat explosion" situation (which could happen if the velocity of the exotermic burning reaction grows sufficiently fast; in this case because of nonexistence of stationary distribution we have fast nonstationary heating of the substance and acceleration of the respective reaction).

It should ne noticed that in the case of $q < 1$ the temperature does not reach stationary state because, cf. Eq. (9), $\langle 1/T \rangle = 1/(T_0 - \varepsilon/\alpha')$, whereas for $q > 1$ we had $< T >= T_0$. As a consequence the corresponding Lévy distribution are defined only for $\varepsilon \in (0, T_0\, \alpha')$ because for $\varepsilon \to T_0\alpha'$ the $< T > \to 0$. Such asymptotic (i.e., for $t/\tau \to \infty$) cooling of the system ($T \to 0$) can be also deduced form Eq. (14) for $\varepsilon \to T_0\alpha'$.

To summarize, we have demonstrated that temperature fluctuations lead to the Lévy distribution $L_q(\varepsilon)$ with index $q < 1$ when there exists energy source and with $q > 1$ in the absence of such source. In both cases, however, the relative variance of $1/T$ fluctuations is described by the parameter q only.

References

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